Journal of Global Optimization **19:** 183–198, 2001.

Exceptional Families and Existence Results for Nonlinear Complementarity Problem

W.T. OBUCHOWSKA
Department of Mathematics, Chowan College, Murfreesboro, NC 27855, USA
(e-mail: obuchw@chowan.edu)

(Received 19 February 1999; accepted in revised form 27 November 2000)

Abstract. In this paper we establish several sufficient conditions for the existence of a solution to the linear and some classes of nonlinear complementarity problems. These conditions involve a notion of the "exceptional family of elements" introduced by Smith [19] and Isac, Bulavski and Kalashnikov [4], where the authors have shown that the nonexistence of the "exceptional family of elements" implies solvability of the complementarity problem. In particular, we establish several sufficient conditions for the nonexistence as well as for the existence of the exceptional family of elements.

Key words: Complementarity problem; Existence of a solution; Exceptional family of elements

1. Introduction

Complementarity theory, which has been studied intensively in the last several decades, is generally considered to be a domain of applied mathematics. The complementarity problem arises in a variety of contexts such as optimization, game theory, economics, classical mechanics, stochastic optimal control, etc. [1, 5, 15]. The primary source of complementarity problems are equilibrium problems in economics, physics and engineering and the necessary conditions for optimality for mathematical programs.

Because of the many important applications of the complementarity problem, the development of the conditions assuring the existence of a solution to this problem was always of big interest. So far many researchers have established a variety of conditions for the solvability of the complementarity problem. These include the existence conditions developed by Eaves [2], Kojima [11], Karamardian [8, 9], Moré [12, 14], Habetler and Price [3], Pang [17] and several other authors.

Our study is motivated by the papers by Smith [20], and Isac, Bulavski and Kalashnikov [4]. Smith introduced a concept of "exceptional sequence" for a continuous function and used it to investigate the conditions for the solution of the complementarity problems. Isac, Bulavski and Kalashnikov extended the results established by Smith to several kinds of complementarity problems proving that the nonexistence of the "exceptional family of elements" implies the solvability of the complementarity problem.

The above result indicates that a definition of the set of conditions under which continuous function does not possess the exceptional family of elements would provide a new practical result in complementarity theory. This problem was investigated in the paper [7]. We proved there that several classes of nonlinear functions (for some of which it is known that the corresponding complementarity problem has a solution), do not have the exceptional family of elements. In this paper we consider a class of continuous functions, which are convex over the convex set $\mathbb{R}^n_+ \setminus D$, where D is a compact set. This class of functions is broader than the class of convex functions on $\mathbb{R}^n_+$, although the most important case is obtained when the functions are linear transformations. We establish several new sufficient conditions for the nonexistence as well as for the existence of the exceptional family of elements. Furthermore, we show how these conditions can be used to determine the existence of a solution to the linear complementarity problem.

The paper is organized as follows. In the next section several sufficient conditions for the existence and nonexistence of the exceptional family of elements for the considered class of the nonlinear complementarity problem are presented. An algorithm to determine the existence of the exceptional family of elements, which is based upon these conditions is proposed in Section 3. A numerical example and results of some computational experiments with the algorithm on several LCP are provided in Section 4. Conclusions are given in the last section.

The following notation is used throughout our paper. The symbols $\mathbb{R}^n$ and $\mathbb{R}^n_+$, denote the $n-$ dimensional Euclidean space and the nonnegative orthant of $\mathbb{R}^n$, respectively. e_j is defined as a vector with the j-th coordinate being 1 and the others being 0. The logic symbols $\vee$, $\wedge$, $\exists$, and $\forall$ are defined according to the standard notation as 'or', 'and', 'exists' and 'for every' correspondingly.

2. Exceptional Family of Elements for Linear and Nonlinear Functions

Let $f : \mathbb{R}^n_+ \to \mathbb{R}^n$, be a continuous mapping.

We will use the abbreviation EFE for the term 'exceptional family of elements for f', which has been defined in [4].

DEFINITION 2.1. [4]. We say that a set of points $\{x^r\}_{r>0} \subset \mathbb{R}^n_+$ is an exceptional family of elements for f with respect to $\mathbb{R}^n_+$, if $\|x^r\| \to +\infty$ as $r \to +\infty$, and for each $r > 0$, there exists $\mu_r > 0$, such that
(i) $f_i(x^r) = -\mu_r x^r_i$, if $x^r_i > 0$,
(ii) $f_i(x^r) \geqslant 0$, if $x^r_i = 0$.

LEMMA 2.1. [4]. *For any continuous mapping $f : \mathbb{R}^n_+ \to \mathbb{R}^n$, there exists either a solution to the nonlinear complementarity problem*

$$NCP(f, \mathbb{R}^n_+) : \textit{ find } x_0 \in \mathbb{R}^n_+ \textit{ such that } f(x_0) \in \mathbb{R}^n_+, \textit{ and } \langle x_0, f(x_0)\rangle = 0,$$

or an exceptional family of elements for f *with respect to* $\mathbb{R}^n_+$.

We will restrict now our considerations only to convex functions.

Let $f : \mathbb{R}^n_+ \to \mathbb{R}^n$, where $f = (f_1, \ldots, f_n)$, and f_i, $i \in I = \{1, 2, \ldots, n\}$, are closed, proper convex functions.

Let $0^+ f_i$ denote the cone of recession of the function f_i, $i = 1, \ldots, n$, i.e. $0^+ f_i$ is a set of all direction vectors along which the function f_i is nonincreasing [18].

LEMMA 2.2. [7]. *Let* $f_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, 2, \ldots, n$ *be convex functions, and* $J_0 = \{i | \mathbb{R}^n_+ \cap 0^+ f_i \neq \emptyset \& \exists x^i \in \mathbb{R}^n_+, f_i(x^i) < 0\}$. *Then*

(i) If $J_0 = \emptyset$ *then there does not exist EFE for* f.

(ii) If there exists such $j \in J_0$, *that*

$$\begin{aligned} e_j \in 0^+ f_j \ \wedge (\forall i \in I \setminus \{j\}, (e_j \notin 0^+ f_i) \vee (e_j \in 0^+ f_i \Rightarrow f_i(te_j) \geqslant 0, \forall t)) \\ \wedge \exists t_0 \in R, f_j(t_0 e_j) < 0 \end{aligned} \tag{1}$$

then there exists an EFE for f.

Lemma 2.2 holds also if the convexity requirement is replaced by a weaker assumption, namely that there exists a compact set D, such that $\mathbb{R}^n_+ \setminus D$ is convex and f_i, $i = 1, 2, \ldots, n$, are convex over $\mathbb{R}^n_+ \setminus D$. The above assumptions on the sets D and $\mathbb{R}^n_+ \setminus D$ remain valid throughout the paper. Generalized version of Lemma 2.2 is given in the Corollary 2.1 below.

COROLLARY 2.1. *Let* $f_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, 2, \ldots, n$ *be convex functions over the convex set* $\mathbb{R}^n_+ \setminus D$. *Let us denote* $I_0 = \{i | \mathbb{R}^n_+ \cap 0^+ f_i \neq \emptyset \& \exists x^i \in \mathbb{R}^n_+ \setminus D, f_i(x^i) < 0\}$. *Then*

(i) If $I_0 = \emptyset$ *then there does not exist EFE for* f.

(ii) If there exists $j \in I_0$, *such that*

$$\begin{aligned} e_j \in 0^+ f_j \ \wedge (\forall i \in I \setminus \{j\}, (e_j \notin 0^+ f_i) \vee (e_j \in 0^+ f_i \Rightarrow f_i(te_j) \geqslant 0, \forall t)) \\ \wedge \exists t_0 \in R, t_0 e_j \in \mathbb{R}^n_+ \setminus D, f_j(t_0 e_j) < 0 \end{aligned} \tag{2}$$

then there exists an EFE for f.

Proof. Since the set I_0 and condition (ii) in Corollary 2.1 are only slight modification of the set J_0 and condition (ii) in Lemma 2.2, the proof of the corollary is similar to the proof of the latter lemma. □

Let $f, g : \mathbb{R}^n \to \mathbb{R}^n$ be continuous mappings, where $f = (f_1, f_2, \ldots, f_n)$ and $g = (g_1, g_2, \ldots, g_n)$. We assume that the functions f_i, g_i, $(i = 1, 2, \ldots, n)$, are convex over $\mathbb{R}^n$.

Let us consider the following (implicit) complementarity problem, introduced in [4].

ICP$(f, g, \mathbb{R}^n_+)$: find $x_0 \in \mathbb{R}^n_+$ such that, $g(x_0) \in \mathbb{R}^n_+$, $f(x_0) \in \mathbb{R}^n_+$ and $\langle g(x_0), f(x_0)\rangle = 0$.

DEFINITION 2.2. [4]. A set of points $\{x^r\}_{r>0} \subset \mathbb{R}^n$ is an exceptional family of elements for the couple (f, g), if $\|x^r\| \to +\infty$ as $r \to +\infty$, $g(x^r) \geqslant 0$ for each $r > 0$ and there exists $\mu_r > 0$ such that for $i = 1, 2, ..., n$

(i) $f_i(x^r) = -\mu_r g_i(x^r)$, if $g_i(x^r) > 0$;
(ii) $f_i(x^r) \geqslant 0$, if $g_i(x^r) = 0$.

LEMMA 2.3. [4]. *Let $f, g : \mathbb{R}^n \to \mathbb{R}^n$ be continuous mappings. If the following assumptions are satisfied:*
(i) the equation $g(x) = 0$ has a unique solution $x = b$,
(ii) g maps a neighborhood of the point b homeomorphically onto a neighborhood of the origin,
then there exists either a solution of the problem $ICP(f, g, \mathbb{R}^n_+)$ or an exceptional family of elements for the couple (f, g).

We will prove in the lemma below the generalization of the Lemma 2.2 to ICP for the couple of functions (f, g) (see definition in [4]), provided that $D = \emptyset$.

Let the symbol $\mathcal{D}^{=}_{g_i}$ denote the constancy space of $g_i(x)$, i.e. $\mathcal{D}^{=}_{g_i} = \{y \in \mathbb{R}^n | y \in 0^+ g_i \wedge -y \in 0^+ g_i\}$ [18].

THEOREM 2.1. *Let $f, g : \mathbb{R}^n \to \mathbb{R}^n$, where $f = (f_1, ..., f_n)$, $g = (g_1, ..., g_n)$, f_i, g_i, $i \in I = \{1, 2, ..., n\}$, be continuous convex functions over $\mathbb{R}^n$. Assume that $g(x)$ satisfies conditions (i) and (ii) of Lemma 2.3.*

Let $I^g_0 = \{i | 0^+ f_i \neq \emptyset \wedge \exists x^i \in \mathbb{R}^n, f_i(x^i) < 0, g_i(x^i) > 0\}$. Then
(i) If $I^g_0 = \emptyset$ then there does not exist an EFE for the couple (f, g) and the $ICP(f, g, \mathbb{R}^n_+)$ problem has a solution.
(ii) If there exists $j \in I^g_0$, such that

$$
\begin{aligned}
& e_j \in 0^+ f_j \wedge \exists t_0 \in \mathbb{R}, t_0 e_j \in \mathbb{R}^n, f_j(t_0 e_j) < 0 \\
& \quad \wedge (e_j \notin 0^+ g_j \vee (\exists t_j, g_j(t e_j) > 0, \forall t \geqslant t_j)) \\
& \quad \wedge (\forall i \in I \setminus \{j\}, \\
& (e_j \notin 0^+ f_i) \vee (e_j \in 0^+ f_i \Rightarrow f_i(t e_j) \geqslant 0, \forall t) \\
& \quad \wedge (e_j \in \mathcal{D}^{=}_{g_i} \wedge \exists \bar{t}_i, \bar{t}_i e_j \in \mathbb{R}^n, g_i(\bar{t}_i e_j) = 0))
\end{aligned}
\tag{3}
$$

then there exists an EFE for the couple (f, g).

Proof. (i) We will prove first that if there does not exist a function f_i with a nonempty cone of recession, such that there exists $x^i \in \mathbb{R}^n$, $f_i(x^i) < 0$, and $g_i(x^i) > 0$, then there does not exist an EFE for (f, g). The proof is by contradiction. Let $\{x^r\}_{r>0}$ be an exceptional family of elements for the couple (f, g). From the assumption (i) of Lemma 2.3 and the convexity of g_i, $i = 1, ..., n$, it follows that there exist indices i and r_0 such that

$$g_i(x^r) > 0, \forall r \geqslant r_0. \tag{4}$$

The definition of the EFE thus implies

$$f_i(x^r) = -\mu_r g_i(x^r) < 0, \quad r \geqslant r_0 \tag{5}$$

with $\|x^r\| \to \infty$. This implies that the level set of f_i is unbounded and consequently $0^+ f_i \neq \emptyset$. Inequalities (4) and (5) with $x^i = x^{r_\tau}$, (where $x^{r_\tau} \in \mathbb{R}^n$), give that $f_i(x^i) < 0$ and $g_i(x^i) > 0$, which implies that $I_0^g \neq \emptyset$. This proves the first part of the lemma.

(ii) Now we will prove that if condition (3) is satisfied, then there exists an exceptional family for f. Let $j \in I_0$ satisfy (3), and let $x(t) = te_j, \quad t \geqslant t_0$. Conditions $e_j \in 0^+ f_j$ and $\exists t_0, t_0 e_j \in \mathbb{R}^n$, $f_j(t_0 e_j) < 0$ imply that

$$f_j(x(t)) < 0, \forall t \geqslant t_0. \tag{6}$$

From the assumption $\forall i \in I \setminus \{j\}, e_j \in \mathcal{D}_{g_i}^{=} \wedge \exists \bar{t}_i, \bar{t}_i e_j \in \mathbb{R}^n, g_i(\bar{t}_i e_j) = 0$ it follows that

$$\forall i, \quad i \neq j, g_i(x(t)) = 0, \forall t. \tag{7}$$

We will prove that

$$\exists t_i, \quad f_i(x(t)) \geqslant 0, \quad \forall t \geqslant t_i. \tag{8}$$

Let us first consider a case when $e_j \notin 0^+ f_i, i \neq j$. Then by convexity f_i is unbounded from above along every line with the direction vector e_j. Therefore (8) holds. On the other hand, if $e_j \in 0^+ f_i$, then by assumption $e_j \in 0^+ f_i \Rightarrow f_i(te_j) \geqslant 0, \ \forall t$, which implies that inequality (8) holds also in this case.

Now, the alternative $e_j \notin 0^+ g_j \vee (\exists t_j, g_j(te_j) > 0, \forall t \geqslant t_j)$ implies that

$$\exists \hat{t}_j, g_j(te_j) > 0, \forall t \geqslant \hat{t}_j. \tag{9}$$

In fact, any unbounded sequence on the half-line $x(t)$ will satisfy conditions of the EFE for (f, g). It can be shown by considering identity

$$f_j(x(t_r)) = \frac{f_j(x(t_r))}{g_j(x(t_r))} g_j(x(t_r))$$

and substituting $x^r = x(t_r)$ and

$$\mu_r = -\frac{f_j(x(t_r))}{g_j(x(t_r))}, \quad t_r > \max\{t_0, \hat{t}_j\}$$

in the expression (i) of the Definition 2.2 of the EFE for the couple (f, g). We note that (6) along with (9), implies $\mu_r > 0$, for $t_r > \max\{t_0, \hat{t}_j\}$. Moreover, relations (7) and (8) assure that conditions (ii) in the definition of the EFE for the couple (f, g) are satisfied. Because relation (8) holds for $t \geqslant t_i$, then finally we choose $t_r > \max\{t_0, \hat{t}_j, t_i, i = 1, ..., n, \ i \neq j, r\}$. Moreover, $\|x^r\| = t_r$, so $\|x^r\| \to \infty$, if $r \to \infty$. □

In Lemma 2.4 and Theorems 2.2, 2.3 and Corollary 2.2 below we assume that $f : \mathbb{R}^n_+ \to \mathbb{R}^n$, where $f = (f_1, f_2, ..., f_n)$ and that all f_i are continuous functions, convex over the convex set $\mathbb{R}^n_+ \setminus D$, where D is a compact set. We first prove the following lemma which will be used in Theorem 2.3 and consequently in the proof of the Algorithm A.

LEMMA 2.4. *Let $f_i : \mathbb{R}^n \to \mathbb{R}$, be convex over the convex set $\mathbb{R}^n \setminus D$. Let $\{x^r\} \in EFE$, and $\{\frac{x^{r_j}}{\|x^{r_j}\|}\}$ denote convergent subsequence of the sequence $\{\frac{x^r}{\|x^r\|}\}$ and $\bar{x}$ will be its limit. For all $i \in \bar{I} = \{i|\bar{x}_i > 0\}$, we have that $\bar{x} \in 0^+ f_i$.*

Proof. The proof will be by contradiction. Suppose that $\bar{x} \notin 0^+ f_i$ for some $i \in \bar{I}$. This implies that $\bar{x}$ is a direction of increase for f_i, over $\mathbb{R}^n_+ \setminus D$, which by Theorem 8.6 in [18] allows us to conclude that f_i is unbounded from above along any half-line with the direction $\bar{x}$.

It follows from $i \in \bar{I}$, that

$$\exists r_0, \forall r_j \geqslant r_0, x_i^r > 0. \tag{10}$$

By Definition 2.1, we have

$$f_i(x^{r_j}) = -\mu_{r_j} x_i^{r_j} < 0, \quad \forall r_j > r_0. \tag{11}$$

Since $\|x^{r_j}\| \to +\infty$, as $r_j \to \infty$ and $\bar{x} \notin 0^+ f_i$, we know that

$$\lim_{r_j \to \infty} f_i(\|x^{r_j}\|\bar{x}) = +\infty.$$

So for arbitrarily large $M > 0$, there exists r_1 such that $\forall r_j > r_1$ we have

$$f_i(\|x^{r_j}\|\bar{x}) > M.$$

By continuity of f_i and $\frac{x^{r_j}}{\|x^{r_j}\|} \to \bar{x}$, we have $\exists \bar{r} \geqslant \max\{r_0, r_1\}$ such that

$$f_i\left(\frac{x^{r_j}}{\|x^{r_j}\|}\|x^{r_j}\|\right) > \frac{M}{2}, \quad \forall r_j > \bar{r},$$

e.g.,

$$f_i(x^{r_j}) > \frac{M}{2},$$

which contradicts to (11).

This ends the proof of the lemma. □

COROLLARY 2.2. *Let $f_i : \mathbb{R}^n \to \mathbb{R}$, be convex over the convex set $\mathbb{R}^n \setminus D$. If for given $i \in I$, and $\{x^r\} \in EFE$, there exists a subsequence $\{x^{r_j}\}$ of $\{x^r\}$ such that $x_i^{r_j} > 0, \forall j$, then for $\bar{x} = \lim_{j\to\infty} \frac{x^{r_j}}{\|x^{r_j}\|}$ we have $\bar{x} \in 0^+ f_i$.*

Proof. Proof follows directly from the observation that in the proof of Lemma 2.4 we based only on the relation (10), not requiring that $\bar{x}_i > 0$. □

THEOREM 2.2. *Let $f_i : \mathbb{R}^n \to \mathbb{R}$, be convex over $\mathbb{R}^n \setminus D$. Let us define $I_0 = \{i|\mathbb{R}^n_+ \cap 0^+ f_i \neq \emptyset \& \exists x^i \in \mathbb{R}^n_+ \setminus D, f_i(x^i) < 0\}$. If the following three assumptions are satisfied:*

(i) $I_0 \neq \emptyset$,

(ii) $\forall j \in I_0$ *the relation in (2) is not satisfied,*
(iii)

$$I_1 = \{(j,k),\, j,k \in I_0,\, j \neq k | \exists s \geqslant 0, s \in 0^+ f_l,\; l = j,k, \exists x_{jk} \in \mathbb{R}^n_+ \setminus D,\\ f_l(x_{jk}) < 0, l = j,k\} = \emptyset,$$

then there does not exist an EFE for f and consequently $NCP(f, \mathbb{R}^n_+)$ problem has a solution.

Proof. Let us assume that assumptions (i)–(iii) are satisfied. We will prove that in this case there does not exist an EFE. The proof will be by contradiction. Under the assumption that $I_1 = \emptyset$, we have that $\forall j,k,\; j \neq k$, either there is no such $s \geqslant 0$, that

$$s \in 0^+ f_j \cap 0^+ f_k,$$

or there does not exist a point $x_{jk} \in \mathbb{R}^n_+ \setminus D$, such that $f_k(x_{jk}) < 0$, $f_j(x_{jk}) < 0$.

Let $\{x^r\}$ be a sequence in EFE for f, where $x^r \in \mathbb{R}^n_+ \setminus D$, and let $\{\bar{x}^r\}$ be such subsequence of the sequence $\{x^r\}$, that for some index j, all j-th coordinates are positive and $\bar{x}^r_j \to \infty$. We will show that for indices l, such that $l \neq j$, there can be only finite number of positive coordinates $\bar{x}^r_l$ in the sequence $\{\bar{x}^r\}$. To this end note that otherwise for some unbounded subsequence $\{\tilde{x}^r\}$ of $\{\bar{x}^r\}$ (corresponding to the components $\bar{x}^r_l > 0$, where $\bar{x}^r = (\bar{x}^r_1, \bar{x}^r_2, ..., \bar{x}^r_n)$), we would have

$$f_l(\tilde{x}^r) = -\mu_r \tilde{x}^r_l < 0 \quad \text{and} \quad f_j(\tilde{x}^r) = -\mu_r \tilde{x}^r_j < 0, \quad \forall r.$$

Because $\|\tilde{x}^r\| \to \infty$, the last two inequalities along with the fact that $\tilde{x}^r \in \mathbb{R}^n_+$ imply that

$$\tilde{x}^r \in \mathbb{R}^n_+ \cap S_l \cap S_j, \quad \text{where } S_l = \{x | f_l(x) < 0\} \quad \text{and} \quad S_j = \{x | f_j(x) < 0\},$$

which proves that $\mathbb{R}^n_+ \cap S_l \cap S_j$ is an unbounded convex set. It follows that $\mathbb{R}^n_+ \cap S_l \cap S_j$ contains a half-line, which gives that there exists s, such that

$$s \in 0^+ f_j \cap 0^+ f_k \cap \mathbb{R}^n_+.$$

In particular any accumulation point of the sequence $\{\frac{\tilde{x}^r}{\|\tilde{x}^r\|}\}$ satisfies the latter relation. Consequently $I_1 \neq \emptyset$, which contradicts the assumption. Thus, it follows that if $I_1 = \emptyset$, then every infinite subsequence of the sequence which has all j-th coordinates positive, has the remaining coordinates equal zero (except possible finite number of elements). Clearly, this subsequence is an EFE for f. Note, that we have $\frac{\bar{x}^r}{\bar{x}^r_j} = e_j, \quad \forall r$ and therefore $e_j \in 0^+ f_j$. Since by assumption (ii) condition (2) does not hold, then

$$\exists k \in I_0, \quad k \neq j, \quad e_j \in 0^+ f_k, \quad \text{and} \quad \exists t_0 > 0, \quad f_k(t_0 e_j) < 0. \tag{12}$$

Therefore

$$e_j \in 0^+ f_j \cap 0^+ f_k,$$

and that $x_j^r x_k^r = 0, \forall r$ (as otherwise for some r_0, $f_j(x^{r_0}) < 0$ and $f_k(x^{r_0}) < 0$ and therefore $I_1 \neq \emptyset$). Because all elements of the sequence $\{\bar{x}_k^r\}$ are zeros, then it follows that the function f_k is nonnegative along $\{\bar{x}^r\}$. On the other hand relation (12) implies that $f_k(te_j) < 0, \quad \forall t > t_0$, which contradicts earlier conclusion that $f_k(te_j) \geqslant 0$, for $t = \bar{x}_j^r, \ \forall r$. This completes the proof of the Theorem 2.2. □

THEOREM 2.3. *Let* $f_i : \mathbb{R}^n \to \mathbb{R}$, *be convex over the convex set* $\mathbb{R}^n \setminus D$. *Let us define* $I_0 = \{i | R_+^n \cap 0^+ f_i \neq \emptyset \& \exists x^i \in \mathbb{R}_+^n \setminus D, f_i(x^i) < 0\}$. *If the following conditions hold:*

(i) $I_0 \neq \emptyset$,

(ii) $\forall j \in I_0$ *the relation in (2) is not satisfied.*

(iii)

$$I_1 = \{(j,k), j,k \in I_0, j \neq k | \exists s \geqslant 0, s \in 0^+ f_l, \ l = j,k, \exists x_{jk} \in R_+^n \setminus D, \\ f_l(x_{jk}) < 0, l = j,k\} \neq \emptyset,$$

(iv)

$$Y_{I_1} = \left\{ y | y \geqslant 0, y \in 0^+ f_\tau, \tau = j,k, \ (j,k) \in I_1 \wedge \lim_{t\to\infty} \frac{y_j f_k(ty)}{t} \right. \\ \left. = \lim_{t\to\infty} \frac{y_k f_j(ty)}{t}, (y_j, y_k) \neq 0) \right\} = \emptyset,$$

then there does not exist an EFE for f *and* $NCP(f, \mathbb{R}_+^n)$ *has a solution.*

Proof. Suppose that it is opposite, namely that there exists $\{x^r\} \in \mathbb{R}_+^n \setminus D$, which belongs to the EFE for f and $Y_{I_1} = \emptyset$. Let $\{x^r\} \in$ EFE, and $\left\{\frac{x^{r_j}}{\|x^{r_j}\|}\right\}$ denote convergent subsequence of the sequence $\left\{\frac{x^r}{\|x^r\|}\right\}$ and $\bar{x}$ be its limit. Let $i \in \bar{I} = \{i | \bar{x}_i > 0\}$.

Let $k \in \{j | j \in \bar{I}\}$ and $\left\{\frac{x^{r_i'}}{\|x^{r_i'}\|}\right\}$ be such subsequence of the sequence $\left\{\frac{x^{r_i}}{\|x^{r_i}\|}\right\}$ that all coordinates $\{x_k^{r_j'}\}$ are strictly positive. Let's proceed with choosing subsequences from the sequence $\left\{\frac{x^{r_i'}}{\|x^{r_i'}\|}\right\}$ so that as a result we will obtain a sequence $\left\{\frac{\tilde{x}^r}{\|\tilde{x}^r\|}\right\}$ where for some set $\tilde{I} \subset I$, we have that $\tilde{x}_\tau^r > 0, \quad \forall r, \forall \tau \in \tilde{I}$, and $x_l^r = 0$, $\forall r, \quad \forall l \notin \tilde{I}$. It follows that the set $\tilde{I}$ has more than one element. Indeed, if $\tilde{I} = \{k\}$, then it follows that condition (2) is satisfied with $j = k$. This however contradicts assumption (ii).

This is clear that $\|\tilde{x}^r\| \to \infty$, with $r \to \infty$ and that $\{\tilde{x}^r\}$ belongs to the EFE for f. Because $\left\{\frac{\tilde{x}^r}{\|\tilde{x}^r\|}\right\}$ is a subsequence of the sequence $\left\{\frac{x^{r_i'}}{\|x^{r_i'}\|}\right\}$ we have that

$$\left\{\frac{\tilde{x}^r}{\|\tilde{x}^r\|}\right\} \to \bar{x}.$$

Let us define the set $\bar{I}_1 = \{i|(i, j) \in I_1 \vee \exists j \in I_1, (j, i) \in I_1\}$. Lemma 2.4 shows that $\bar{x} \in 0^+ f_i, \forall i \in \bar{I}$ and from the Corollary 2.2, we get that $\bar{x} \in 0^+ f_i, \forall i \in \tilde{I} \subset \bar{I}_1$, and that $\bar{I} \subset \tilde{I} \subset \bar{I}_1$. Because $Y_{I_1} = \emptyset$ and $I_1 \neq \emptyset$, we have that for (j, k), where $j, k \in \tilde{I}$

$$\lim_{t\to\infty} \frac{\bar{x}_j f_k(t\bar{x})}{t} \neq \lim_{t\to\infty} \frac{\bar{x}_k f_j(t\bar{x})}{t}, \tag{13}$$

On the other hand, because the sequence $\{\tilde{x}^r\}$ belongs to the exceptional family of elements and indices j and k belong to $\tilde{I}$, then

$$f_j(\tilde{x}^r) = -\mu_r \tilde{x}^r_j,$$
$$f_k(\tilde{x}^r) = -\mu_r \tilde{x}^r_k,$$

which implies that

$$\frac{\tilde{x}^r_j}{\|\tilde{x}^r\|}(f_k(\tilde{x}^r)) - \frac{\tilde{x}^r_k}{\|\tilde{x}^r\|}(f_j(\tilde{x}^r)) = 0, \quad \forall r.$$

Dividing both sides of the latter equation by $\|\tilde{x}^r\|$ and taking the limit of the left-side of the equation, with $r \to \infty$, yields

$$\lim_{r\to\infty} \frac{\tilde{x}^r_j}{\|\tilde{x}^r\|}\left(\frac{f_k(\tilde{x}^r)}{\|\tilde{x}^r\|}\right) - \lim_{r\to\infty}\left(\frac{\tilde{x}^r_k}{\|\tilde{x}^r\|}\frac{f_j(\tilde{x}^r)}{\|\tilde{x}^r\|}\right) = 0, \quad \forall r. \tag{14}$$

Given that $\|\tilde{x}^r\| \to \infty$ with $r \to \infty$ and that $\lim_{r\to\infty} \frac{\tilde{x}^r_\tau}{\|\tilde{x}^r\|} = \bar{x}_\tau$, $\tau = j, k$ it follows that the equation (14) contradicts the inequality (13). This completes the proof of the theorem. □

In the LCP important role play so called P-matrices and S-matrices (and closely related to them P_0 and S_0-matrices, respectively) [1,10,13,14]. The theorem below shows some relationship between the S-matrices and the set I_0. We recall that A is an S-matrix if, and only if, the linear complementarity problem

$$x \geqslant 0, \quad Ax - b \geqslant 0, \quad x^T(Ax - b) = 0$$

is feasible for all $b \in \mathbb{R}^n$. Equivalently, $A \in \mathbb{R}^{n\times n}$ is an S-matrix if there is an $x \neq 0$, such that $x \geqslant 0$ and $Ax > 0$, while $A \in R^{n\times n}$ is an S_0-matrix if there is an $x \neq 0$, such that $x \geqslant 0$ and $Ax \geqslant 0$.

THEOREM 2.4. *(i) If $I_0 = \emptyset$ and $b > 0$, then A is an S-matrix.*

(ii) If $I_0 = \emptyset$ then A is an S_0-matrix.

Proof. If $I_0 = \emptyset$ and $b > 0$, then $a_i^T s > 0$ for every $s \geqslant 0$ and every $i \in I$, which implies that the system $As > 0$, $s \geqslant 0$ has a solution. Thus it follows from the definition that A is an $S-$matrix. To show the part (ii) it is enough to note that if $I_0 = \emptyset$, then $a_i^T s \geqslant 0$ for every $s \geqslant 0$, $i \in I$, which implies that the system $As \geqslant 0$, $s \geqslant 0$ has a nonzero solution. Therefore A is an S_0-matrix. □

The theorem below provides some properties of the $P-$functions related to the cone of recession of f_i, $i \in I$. Let $\mathbb{R}^n_{++}$ denote the set of all $x \in \mathbb{R}^n$ with $x > 0$. We recall that f_i, $i \in I$ is called faithfully convex [19], if it is affine on a line segment only if it is affine on the whole line containing that segment.

THEOREM 2.5. *(i) If a function* $f : \mathbb{R}^n_+ \to \mathbb{R}^n$ *is a P-function and* f_i, $i = 1, \ldots, n$ *are convex over convex set* $\mathbb{R}^n_+ \setminus D$*, then*

$$B = \mathbb{R}^n_+ \cap \bigcap_{i=1}^{n} 0^+ f_i = \emptyset.$$

Consequently $\forall i \in I, \exists j \in I$*, such that* $e_i \notin 0^+ f_j$.

(ii) If a function $f : \mathbb{R}^n_+ \to \mathbb{R}^n$ *is a* P_0-*function and* f_i, $i = 1, \ldots, n$ *are faithfully convex over convex set* $\mathbb{R}^n_+ \setminus D$*, then*

$$B_0 = \mathbb{R}^n_{++} \cap \bigcap_{i=1}^{n} (0^+ f_i \setminus D^{=}_{f_i}) = \emptyset.$$

Proof. (i) Let us suppose that the opposite is true, that is a function f (convex on $\mathbb{R}^n_+ \setminus D$), is a P-function and $B \neq \emptyset$. Since $\mathbb{R}^n_+ \setminus D$ is fulldimensional, then for any $s \in B$ there exist $x, y \in \mathbb{R}^n_+ \setminus D$, such that $s = x - y$. Thus $x - y \in \mathbb{R}^n_+$, and consequently $x_i \geqslant y_i$, $\forall i$. From $x - y \in 0^+ f_i$, and convexity of f_i over $\mathbb{R}^n_+ \setminus D$ it follows that $f_i(x) \leqslant f_i(y)$, $\forall i$. Therefore,

$$(y_i - x_i)(f_i(y) - f_i(x)) \leqslant 0,$$

for all i, which contradicts the definition of P-function. The second statement in part (i) of the theorem follows directly.

(ii) If f is a P_0-function and $B_0 \neq \emptyset$ then for $s \in B_0$ there exist $x, y \in \mathbb{R}^n_+ \setminus D$, such that $s = x - y$. Thus $x - y \in \mathbb{R}^n_{++}$, which implies that $x_i > y_i$, $\forall i$. From $x - y \in \bigcap_{i=1}^{n} (0^+ f_i \setminus D^{=}_{f_i})$ and the assumption that f_i is a faithfully convex on $\mathbb{R}^n_+ \setminus D$ it follows that the function is strictly decreasing along every half-line with the direction vector $y - x$. Thus $f_i(y) > f_i(x)$, $\forall i$, and consequently

$$(y_i - x_i)(f_i(y) - f_i(x)) < 0,$$

which contradicts the definition of the P_0-function. □

3. An Algorithm to Determine the Existence and Nonexistence of the EFE for the Linear Transform

The results obtained in the Lemmas 2.2, 2.4, and Theorems 2.2 and 2.3 not only provide new theoretical results on the existence of the solution to the linear and some class of nonlinear complementarity problems but, as demonstrated below, they can be used in the form of the algorithm to determine the existence of the exceptional family of elements for the affine transformation $f(x) = Ax - b$, where $A \in \mathbb{R}^{n\times n}$, $b \in \mathbb{R}^n$.

Now we will restrict our considerations only to the linear functions, that is $f_i(x) = a_i^T x - b_i$, where $a_i \in \mathbb{R}^n$, $b_i \in \mathbb{R}$, for $i = 1, 2, ..., n$.

Thus we consider the linear complementarity problem defined as:

$$LCP(Ax - b, \mathbb{R}^n_+) : \text{find } x_0 \geqslant 0 \text{ such that } Ax_0 - b \geqslant 0 \text{ and } x_0^T(Ax_0 - b) = 0,$$

where the matrix $A^T = [a_{ij}] \in \mathbb{R}^{n\times n}$ consists of the column vectors a_i, $i = 1, ...n$, and $b = [b_i] \in \mathbb{R}^n$.

We also assume that $D = \emptyset$. In this case $0^+ f_i = \{s \in \mathbb{R}^n | a_i^T s \leqslant 0\}$ and the set I_0 defined in previous section has a form $I_0 = \{i | \exists s \geqslant 0, s \neq 0,\ a_i^T s \leqslant 0, \exists x_0^i \in \mathbb{R}^n_+,\ a_i^T x_0^i < b_i\}$

Let us define $e_{jk} = \alpha_j e_j + \alpha_k e_k$, $\alpha_j, \alpha_k > 0$, and $e'_{jk} = \alpha'_j e_j + \alpha'_k e_k$, where $\alpha'_l > 0$, $l = j, k$.

The following algorithm provides sequence of sufficient conditions for the existence and nonexistence of the EFE for the linear transforms, which by Lemma 2.1 are sufficient conditions for the existence of the solution to the $LCP(Ax - b, \mathbb{R}^n_+)$ problem.

ALGORITHM A

Step 1. If $I_0 = \emptyset$ then there does not exist EFE for f (and consequently $LCP(Ax - b, \mathbb{R}^n_+)$ has a solution). If there exists $j \in I_0$, such that

$$a_j^T e_j \leqslant 0,\ a_i^T e_j \geqslant 0,\ i \neq j,\ \exists t_0 \in \mathbb{R},\ a_j^T t_0 e_j < b_j, a_i^T e_j = 0 \Rightarrow b_i \leqslant 0 \tag{A1}$$

then there exists an EFE for f.

Step 2. Consider the set

$$\begin{aligned} I_1 &= \{(j, k), j, k \in I_0, j \neq k | \exists s \geqslant 0, a_l^T s \leqslant 0, l \\ &= j, k, \exists x_{jk} \in \mathbb{R}^n_+, a_l^T x_{jk} < b_l, l = j, k\}. \end{aligned}$$

If $I_1 = \emptyset$ then there does not exist an EFE for f (and consequently solution to $LCP(Ax - b, \mathbb{R}^n_+)$ exists).

Let

$$Y_{I_1} = \{y | y \geqslant 0, a_\tau^T y \leqslant 0, \tau = j, k, \;\; (j,k) \in I_1 \wedge y_j a_k^T y = y_k a_j^T y \wedge (y_j, y_k) \neq 0\}.$$

If $Y_{I_1} = \emptyset$ then there does not exist an EFE, which implies that the solution to $LCP(Ax - b, \mathbb{R}^n_+)$ exists.

Consider all pairs (j,k), where $(j,k) \in I_1$, such that for some $\alpha_l > 0$, $\alpha_l' >$, $l = k, j$, the vectors $e_{jk} = \alpha_j e_j + \alpha_k e_k$, $e_{jk}' = \alpha_j' e_j + \alpha_k' e_k$ satisfy

$$a_\tau^T e_{jk} \leqslant 0, \tau = j, k \wedge (\forall i \in I \setminus \{j,k\}, a_i^T e_{jk} \geqslant 0), a_\tau^T e_{jk}' < b_\tau, \tau = j, k, (a_i^T e_{jk} = 0 \Rightarrow a_i^T e_{jk}' \geqslant b_i). \tag{A2}$$

Let S_1 denote the set of vectors e_{jk} satisfying (A2). If $S_1 = \emptyset$ then go to Step 3.

If $S_1 \neq \emptyset$ then verify whether there exists a vector $y = e_{jk}$ in the set $Y_{I_1} \cap S_1$, such that the equation

$$y_j(a_k^T ty - b_k) = y_k(a_j^T ty - b_j), \quad \forall t \geqslant 0 \tag{A3}$$

(where $y = (y_1, y_2, ..., y_n)$), is satisfied and if there exists $\bar{x}_{jk} = e_{jk}'$ satisfying

$$\bar{x}_{jk}^j(a_k^T \bar{x}_{jk} - b_k) = \bar{x}_{jk}^k(a_j^T \bar{x}_{jk} - b_j) \tag{A4}$$

and $a_l^T \bar{x}_{jk} < b_l$, $l = j, k$. If such a pair (j,k) exists then there exists an EFE for f.

Step 3. Consider the set

$$I_2 = \{(j,k,l) | (j,k), (k,l), (j,l) \in I_1, j \neq k \neq l, \exists s \geqslant 0, a_\tau^T s \leqslant 0, \tau = j, k, l, \exists x_{jkl} \in \mathbb{R}^n_+, a_\tau^T x_{jkl} < b_\tau, \tau = j, k, l\}.$$

If $I_2 = \emptyset$ then there does not exist an EFE and $LCP(Ax - b, \mathbb{R}^n_+)$ has a solution.

Let us define

$$Y_{I_2} = \{y | y \geqslant 0, a_\tau^T y \leqslant 0, \tau = j, k, l, (j,k,l) \in I_2 \wedge (y_j a_\tau^T y = y_\tau a_j^T y, \tau = k, l, \;\; y_k a_l^T y = y_l a_k^T y) \wedge (y_j, y_k, y_l) \neq 0\}.$$

If $Y_{I_2} = \emptyset$ then there does not exist an EFE and $LCP(Ax - b, \mathbb{R}^n_+)$ is solvable. □

We omit the proof of the algorithm for Step 1, and for this part of the Step 2 which involves checking whether or not the sets I_1 and Y_{I_1} are empty. These parts of the proof follow directly from Lemma 2.2 and Theorems 2.2 and 2.3 with $f_i(x) = a_i^T x - b_i$.

In the next theorem we give the proof of the remainder of the Step 2 of the algorithm. The proof of the Step 3 is analogous to the proof of the first part of the Step 2.

THEOREM 3.1. *If Algorithm A terminates in Step 2 with $S_1 \neq \emptyset$ and the equations (A3)–(A4) are satisfied, then there exists an EFE for f.*

Proof. Suppose that the vector $y = e_{jk} \in Y_{I_1} \cap S_1$ and satisfies conditions (A3) and $\bar{x}_{jk} = e'_{jk}$ satisfies condition (A4). Condition (A2) assures that the line $x(t) = \bar{x}_{jk} + ty, \quad t \geqslant 0$, satisfies $a_j^T x(t) < b_j$ and $a_k^T x(t) < b_k, \quad \forall t \geqslant 0$ and

$$\forall i, \quad i \neq j, k, \quad \exists t_i, \quad a_i^T x(t) \geqslant b_i, \quad \forall t \geqslant t_i.$$

The latter inequality is satisfied because the vector e_{jk} satisfies the inequality $a_i^T e_{jk} \geqslant 0$, along with the implication ($a_i^T e_{jk} = 0 \Rightarrow a_i^T e'_{jk} \geqslant b_i, \forall i \neq j, k$.)

We will now show that any sequence $\{x^r\}$ of points lying on the line $x(t) = e'_{jk} + te_{jk}, \quad t \geqslant t_0$ belongs to the EFE.

Note that the condition (A3) assures that the vector e_{jk} is a direction of constancy for the function

$$G_{jk}(x) = x_j(a_k^T x - b_k) - x_k(a_j^T x - b_j).$$

This means that the function $G_{jk}(x)$ is constant along the half-line $x(t) = e'_{jk} + te_{jk}, t \geqslant t_0$, i.e. $G_{jk}(x(t)) = C, \forall t$, where C is a constant number. From equality (A4) it follows that $C = 0$. Now, equality $G_{jk}(x(t)) = 0$, implies that

$$\frac{f_j(x(t_r))}{(\alpha'_j + t_r \alpha_j)} = \frac{f_k(x(t_r))}{(\alpha'_k + t_r \alpha_k)}.$$

Let's write

$$f_j(x(t_r)) = \frac{a_j^T x(t_r) - b_j}{(\alpha'_j + t_r \alpha_j)}(\alpha'_j + t_r \alpha_j), \tag{16}$$

and

$$f_k(x(t_r)) = \frac{a_k^T x(t_r) - b_k}{(\alpha'_k + t_r \alpha_k)}(\alpha'_k + t_r \alpha_k). \tag{17}$$

Substitution of

$$\frac{f_k(x(t_r))}{(\alpha'_k + t_r \alpha_k)} = -\mu_r$$

and $x_k^r = \alpha'_k + t_r \alpha_k$ and $x_j^r = \alpha'_j + t_r \alpha_j$ in (16)–(17) proves that any unbounded sequence on the line $x(t)$ belongs to the EFE for f.

This completes the proof of the theorem. □

Algorithm A can be directly extended to the higher number of steps, involving $m-$touples $(j_1, ..., j_m)$ in the $m-$th step. We note that although such an algorithm would terminate in at most n iterations, the implementation of further steps would require further study.

4. Examples

Algorithm A has been tested on various LCP problems studied in the literature. In particular we investigated the performance of the algorithm on most of the problems in [1], as well as on two problems given in [6, 16]. Because all tested problems were low dimensional, the algorithm terminated in either the first, second or very rarely in the third step.

We illustrate the algorithm on some of the tested problems.

EXAMPLE 4.1 (Problem 4.6.4 in [1]). *Consider the following LCP problem which has a solution. The matrix is positive, strictly semimonotone, although it is not a P-matrix.*

$$A = \begin{pmatrix} 2 & 3 & 3 & 2 \\ 2 & 2 & 2 & 3 \\ 2 & 3 & 3 & 1 \\ 1 & 1 & 1 & 2 \end{pmatrix}, \quad b = (10, 12, 9, 8)^T.$$

It follows immediately that $I_0 = \emptyset$, which indicates that the corresponding LCP has a solution. Because $I_0 = \emptyset$ for every b then the LCP is solvable for every value of b.

EXAMPLE 4.2 (Problem 4.11.8 in [1]). *This problem has been used to demonstrate Murty's last index method.*

$$A = \begin{pmatrix} .2 & 0 & .2 \\ .2 & .1 & 0 \\ 0 & .2 & .1 \end{pmatrix}, \quad b = (1, 1, 1)^T.$$

Step 1. It is easily seen that $I_0 = I$. Conditions (A1) are not satisfied for any $j \in I_0$.
Step 2. It follows immediately that $I_1 = \emptyset$, which shows that LCP has a solution. Moreover, it has a solution for every positive value of b, because the outcome of both steps of the algorithm does not depend on b for this matrix as long as $b > 0$.

EXAMPLE 4.3 (Problem 5.1 in [6]). *Consider the following problem which is feasible but not solvable*

$$A = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}, \quad b = (2, 1)^T.$$

The Step 1 gives $I_0 = \{1\}$. The set of conditions (A1) is clearly satisfied by the vector e_1, which shows that there exists EFE. Also for this problem, the outcome of the Algorithm would be the same for arbitrary vector b.

EXAMPLE 4.4 (Problem 4.11.13 in [1]). *Consider LCP in which*

$$A = \begin{pmatrix} 21 & 0 & 0 \\ 28 & 14 & 0 \\ 24 & 24 & 12 \end{pmatrix}, \quad b = (1, 1, 1)^T.$$

Step 1. It is easily seen that $I_0 = \{1, 2\} \neq \emptyset$. *There is no* $j \in I_0$ *satisfying conditions (A1).*
Step 2. It follows immediately that $I_1 = \{(1, 2)\} \neq \emptyset$, *and that* $Y_{I_1} = \emptyset$. *The latter relation follows from the fact that the only solution to the homogeneous system corresponding to the first two rows of the matrix is* $y = (0, 0, y_3)$, $y_3 > 0$. *Therefore, the LCP has a solution. It is clear that the outcome of the algorithm will be the same for arbitrary vector* $b = (b_1, b_2, b_3)$, *with* $b_1 > 0,\ b_2 > 0$.

5. Final Remarks

The performance of the algorithm has been tested on number of LCP problems. The solution of these problems required at most three steps of the algorithm, which may be due to the fact they were either low dimensional or the matrix A had a special structure (e.g. A was a sparse matrix), assuring that either the number of elements in the sets I_0 or I_1 was small or the conditions occuring in Steps 2 and 3 were easy to check. Implementation of higher steps (including this part of the Step 2 which involves the set S_1 and relations (A3) and (A4)), requires further study. Relatively straightforward is the implementation of checking whether or not the set I_{m-1} is empty.

Another open problem is to generalize results stated in Theorems 2.2, 2.3 and 3.1 to the $ICP(f, g, \mathbb{R}^n_+)$ problem.

Acknowledgement

The author wishes to thank the anonymous referees for their suggestions which have helped to greatly improve this paper.

References

1. Cottle, R.W., Pang, J.S., and Stone, R.E. (1992), *The Linear Complementarity Problem*, Academic Press, New York.
2. Eaves, B.C. (1971), On the Basic Theorem of Complementarity, *Mathematical Programming* 1: 68–75.
3. Habetler, G.J., and Price, A.L. (1971), Existence Theorem for Generalized Nonlinear Complementarity Problems, *Journal of Optimization Theory and Applications* 4: 223–239.
4. Isac, G., Bulavski V. and Kalashnikov, V. (1997), Exceptional Families, Topological Degree and Complementarity Problems, *Journal of Global Optimization* 10: 207–225.
5. Isac, G. (1992), Complementarity Problems, *Lecture Notes in Mathematics*, Springer-Verlag, No. 1528.
6. G. Isac, G., Kostreva, M.M. and Wiecek, M.M. (1995), Multiple-Objective Approximation of Feasible but Unsolvable Linear Complementarity Problems, *Journal of Optimization Theory and Applications* 86: 389–405.
7. Isac, G., and Obuchowska, W.T. (1998), Functions Without Exceptional Family of Elements and Complementarity Problems, *Journal of Optimization Theory and Applications* 99: 147–163.

8. Karamardian, S. (1976), An Existence Theorem for the Complementarity Problem, *Journal of Optimization Theory and Applications* 19: 227–232.
9. Karamardian, S. (1971), Generalized Complementarity Problem, *Journal of Optimization Theory and Applications* 8: 161–168.
10. Kojima, M., Mizuno, S., and Noma, T. (1989), A New Continuation Method for Complementarity Problems with Uniform P-functions, *Mathematical Programming* 43: 107–113.
11. Kojima, M. (1975), A Unification of the Existence Theorems of the Nonlinear Complementarity Problem, *Mathematical Programming* 9: 257–277.
12. Moré, J.J. (1974), Coercivity Conditions in Nonlinear Complementarity Problem, *SIAM Reviews* 16: 1–16.
13. Moré, J.J., and Rheinboldt, W. (1973), On P- and S-functions and Related Classes of n-dimensional Nonlinear Mappings, *Linear Algebra and its Applications* 6: 45–68.
14. Moré, J.J. (1996), Global Methods for Nonlinear Complementarity Problems, *Math. of Oper. Research* 21(3): 589–614.
15. Murty, K.G. (1987), *Linear Complementarity, Linear and Nonlinear Programming*, Heldermann Verlag, West Berlin.
16. Pang, J.S. (1991), Iterative Descent Algorithms for a Row Sufficient Linear Complementarity Problem, *SIAM Journal on Matrix Analysis and Applications* 12: 611–624.
17. Pang, J.S., and Yao, J.C. (1995), On Generalization of a Normal Map and Equation, *SIAM J. Control Optimization* 33: 168–184.
18. Rockafellar, R.T. (1970), *Convex Analysis*, Princeton University Press, Princeton, New Jersey.
19. Rockafellar, R.T. (1970), Some Convex Programs Where Duals Are Linearly Constrained, in: J.B. Rosen, O.L. Mangasarian and K. Ritter (eds), pp. 293–322, *Nonlinear Programming*, Academic Press, New York, New York.
20. Smith, T.E. (1984), A Solution Condition for Complementarity Problems with an Application to Special Price Equilibrium, *Applied Mathematics and Computation* 15: 61–69.